Efficient AC Optimal Power Flow & Global Optimizer Solutions

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August 6, 2013
CERTS R&M Review Meeting

Thanks to DOE CERTS for funding

Contents

- Objective
- Our QCQP model
- Global solution
- Efficient algorithm

Objective

Development of an algorithm

- Efficient to solve AC OPF for a large-scale system
- Seeking for the global optimizer

AC Optimal Power Flow

- Find an optimal solution to meet all the economic, operational, and engineering constraints in power system operation
- Computationally complex due to its non-convexity, nonlinearity, and large-scale
- Needs to be solved in a timely manner
  - Weekly in 8hrs, Daily in 2hrs, ***<u>Hourly in 15mins</u>***
  - Each 5mins in 1min, Self-healing post-contingency 0.5 mins

Challenges to Efficient Algorithm

- Non-convexity: May not be solve reliably and efficiently
- Nonlinearity: High cost of computation in Newton update per iteration
- Large-scale network
  - Bus related variables:
    Real and imaginary components of voltage
  - Generator related variables:
    - Real power generation
    - Reactive power generation
    - Cost variable

Algorithm to Solve AC OPF

- Voltage is a phasor → Polar Coordinate System
- Power flow equations involve sinusoidal functions
- MATPOWER: Primal-dual interior point method
- In an NR update, the evaluation and the factorization of the Hessian matrix of Lagrangian need to be performed
  - The factorization number for determining 15-minute dispatches over 30 years is about 11 million for the same transmission network

Recent Approaches

AC OPF in the Cartesian coordinate system

- AC OPF becomes a nonconvex QCQP with quartet flow constraints
- Non-convexity lies in
  - Power balance equality constraints
  - Minimum voltage magnitude constraints
- Commonly used technique: Rank relaxation
  - Convex optimization
  - Easy to solve and yields the global solution
- Zero duality gap under the assumption on the rank
  - Many cases observed with rank > 2

  → Not a physically meaningful solution

  → Lower bound for AC OPF

  → Branch-and-bound method for finding the global optimizer

Inputs & Variables of AC OPF

- Inputs
  - *Φ*: indefinite matrices with real and reactive power balance equations
  - *W*: matrices with voltage magnitudes
  - *Π*: matrices associated with $|i|^2$ and *v*
  - *d*: real and reactive power loads
  - Upper and lower bounds
- Variables: $3N_G + 2N_B \sim \vartheta(N_B)$
  - *v*: real and imaginary components of voltage
  - *p, q*: real and reactive power generation
  - *ξ*: cost variables

Nonconvex AC OPF

- Indices
  - Bus index, *j*
  - Line index, *m*
- Nonconexity
  - *Φ*'s are indefinite matrices
  - Minimum voltage magnitude
- Quartet
  - Flow limits are quartet

$$\min_{v,p,q,\xi} 1^T \xi$$

$$s.t. \begin{cases} v^T \Phi_j^p v - l_j\, p + d_j^p = 0, \ \forall\, j \\ v^T \Phi_j^q v - l_j\, q + d_j^q = 0, \ \forall\, j \\ \underline{|v_j|}^2 \le v^T W_j v \le \overline{|v_j|}^2, \ \forall\, j \\ \left|v^{m,f}\right|_2^2 \ v^T \Pi_f^m v \le f_m, \ \forall\, m \\ \left|v^{m,t}\right|_2^2 \ v^T \Pi_t^m v \le f_m, \ \forall\, m \\ \underline{p} \le p \le \overline{p} \\ \underline{q} \le q \le \overline{q} \\ A_p p + A_\xi \xi \le bc \end{cases}$$

Challenges to Efficient Algorithm

- Non-convexity:

  May not be solve reliably and efficiently
- Nonlinearity
- Large-scale network

Convexification of AC OPF

- Regularization
  - Power balance equation
  - Minimum voltage magnitude
- Additional terms
  - Vanishes quadratically as converges ($v_k - v_{k-1} \rightarrow 0$)
  - At the termination of the algorithm, the approximated constraints is identical to the original constraints

→ Convex constraints

$$v^T \Phi_j^p v - l_j\, p + d_j^p = 0$$

$$\rightarrow v_k^T \Phi_j^p v_k - l_j\, p_k + d_j^p + \lambda_{j,p}^- \left\| \left(\phi_j^{p-}\right)^T \left(v_k - v_{k-1}\right) \right\|_2^2 = 0$$

$$v^T \Phi_j^q v - l_j\, q + d_j^q = 0$$

$$\rightarrow v_k^T \Phi_j^q v_k - l_j\, q_k + d_j^q + \lambda_{j,q}^- \left\| \left(\phi_j^{q-}\right)^T \left(v_k - v_{k-1}\right) \right\|_2^2 = 0$$

$$\text{where } \Phi_j^p = \begin{bmatrix} \left(\phi_j^{p+}\right)^T \\ \left(\phi_j^{p-}\right)^T \\ \left(\phi_j^{p0}\right)^T \end{bmatrix}^T \begin{bmatrix} \lambda_{j,p}^+ I_2 & & \\ & -\lambda_{j,p}^- I_2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} \left(\phi_j^{p+}\right)^T \\ \left(\phi_j^{p-}\right)^T \\ \left(\phi_j^{p0}\right)^T \end{bmatrix}$$

$$= \lambda_{j,p}^+ \phi_j^{p+} \left(\phi_j^{p+}\right)^T - \lambda_{j,p}^- \phi_j^{p-} \left(\phi_j^{p-}\right)^T, \ \lambda_{j,p}^+ > 0, \lambda_{j,p}^- > 0$$

$$\underline{\left|v_j\right|}^2 \le v^T W_j v, \ W_j = \begin{pmatrix} e_j & e_{N_B+j} \end{pmatrix} \begin{pmatrix} e_j & e_{N_B+j} \end{pmatrix}^T$$

$$\rightarrow \underline{\left|v_j\right|}^2 + \left\| \begin{pmatrix} e_j & e_{N_B+j} \end{pmatrix}^T \left(v_k - v_{k-1}\right) \right\|_2^2 \le v^T W_j v$$

Quadratic Approximation to Flow Limits

$$v_k^T \Pi_f^m v_k \leq \frac{f_m}{\left| v_{k-1}^{m,f} \right|_2^2} \quad vs. \quad v_k^T \overline{\Pi_f^m} v_k \leq f_m \left[\left(\frac{\left| v_{k-1}^{m,f} \right|}{v_{ref}} \right)^2 + \left(\frac{v_{ref}}{\left| v_{k-1}^{m,f} \right|} \right)^2 \right]$$

- Leading term in the difference: $\vartheta\left(\left\| v_k - v_{k-1} \right\|_2^2 \right)$
  - As the solution converges, the error vanishes quadratically
  - At a solution, two constraints are identical
- Problem becomes convex QCQP

Convex QCQP

At the k^{th} iteration, a convex QCQP problem is formulated to approximate AC OPF

- Convex relaxation with regularization
- Quartet flow limits are approximated with QC
- As the solution converges, the error vanishes
- Semi-definite programming or reformulation-linearization technique

$$\min_{v_k,\, p_k,\, q_k,\, \xi_k} \; 1^T \xi_k : \quad \text{subject to}$$

$$v_k^T \boxed{\left[\left(\phi_j^{p+}\right)^T \phi_j^{p+}\right]} v_k - \frac{2\lambda_{j,p}^-}{\lambda_{j,p}^+}\left[\left(\phi_j^{p-}\right)^T \phi_j^{p-} v_{k-1}\right]^T v_k$$

$$-\left(\frac{l_j}{\lambda_{j,p}^+}\right) p_k + \frac{1}{\lambda_{j,p}^+}\left(d_j^p + \lambda_{j,p}^- \left\|\phi_j^{p-} v_{k-1}\right\|_2^2\right) = 0, \; \forall\, j$$

$$v_k^T \boxed{\left[\left(\phi_j^{q+}\right)^T \phi_j^{q+}\right]} v_k - \frac{2\lambda_{j,q}^-}{\lambda_{j,q}^+}\left[\left(\phi_j^{q-}\right)^T \phi_j^{q-} v_{k-1}\right]^T v_k$$

$$-\left(\frac{l_j}{\lambda_{j,q}^+}\right) q_k + \frac{1}{\lambda_{j,q}^+}\left(d_j^q + \lambda_{j,q}^- \left\|\phi_j^{q-} v_{k-1}\right\|_2^2\right) = 0, \; \forall\, j$$

$$v_k^T W_j v_k \le \overline{\left|v_j\right|}^2, \; \forall\, j$$

$$\frac{1}{2}\left(\underline{\left|v_j\right|}^2 + v_{k-1}^T W_j v_{k-1}\right) \le \left(W_j v_{k-1}\right)^T v_k, \; \forall\, j$$

$$v_k^T \overline{\Pi_f^m} v_k \le f_m \left[\frac{\left|v_{k-1}^{m,f}\right|_2^2}{v_{ref}^2} + \frac{v_{ref}^2}{\left|v_{k-1}^{m,f}\right|_2^2}\right], \; \forall\, m$$

$$v_k^T \overline{\Pi_t^m} v_k \le f_m \left[\frac{\left|v_{k-1}^{m,t}\right|_2^2}{v_{ref}^2} + \frac{v_{ref}^2}{\left|v_{k-1}^{m,t}\right|_2^2}\right], \; \forall\, m$$

$$\underline{p} \le p_k \le \overline{p}; \quad \underline{q} \le q_k \le \overline{q}; \quad A_p p_k + A_\xi \xi_k \le bc$$

Challenges to Efficient Algorithm

- Non-convexity → Sequential convexification
- Nonlinearity: High cost of computation in Newton update per iteration
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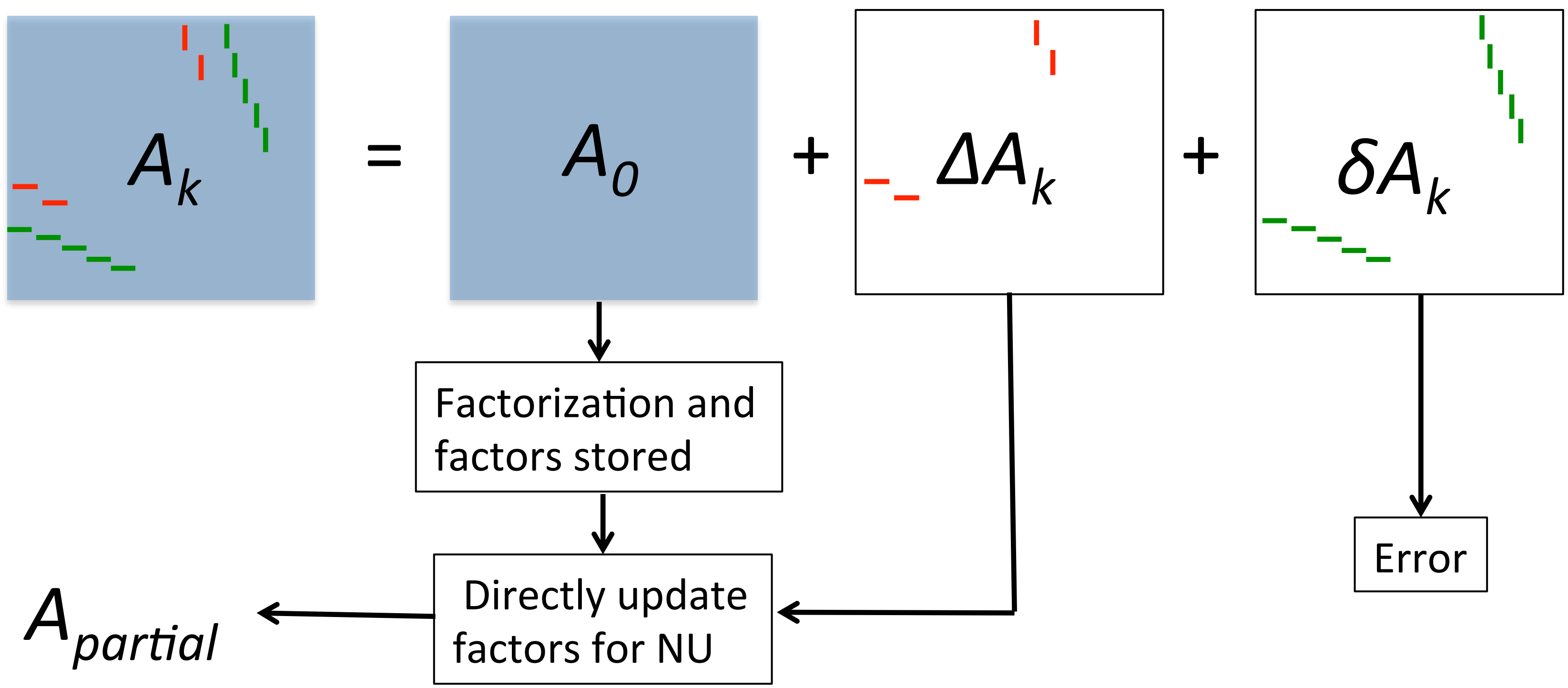
Partial Update I

Lagrange relaxation for the trust region method

- In the Newton update, solve $A_k x_k = b_k$
- At the k^{th} iteration, $A_k = A_0 + \Delta A_k + \delta A_k$
  - A_0 is fixed with a given network: no repeated update or factorization required
  - ΔA_k is significantly large enough to affect x_k
  - δA_k is very small
  - To recover the exact A_k and accordingly x_k, the computational cost is exactly same as the process with A_k
  - $A_0 + \Delta A_k$ is a good approximation to A_k

Partial Update II

Idea: If Hessian is not a rapidly varying matrix, factors are stored for reuse after partial update



Partial Update III

Factorization of A_k where $A_k = A_0 + \Delta A_k + \delta A_k$

- All A's are sparse
- A_0 : sparse factorization performed once and stored for 11 million times reuse
- ΔA_k: determined each iteration, and used to directly update the factors of A_0
- The choice of ΔA_k dictates the efficiency of the partial update
  - Low computation cost to update factors
  - y_k to $(A_0 + \Delta A_k)\, y_k = b_k$ is a good approximation to x_k
- δA_k: modeled as an error

Total Least Square Problem

- The problem is modeled as $(A_0 + \Delta A_k)\, y_k = b_0$
  - δA_k and δb_k $(= b_k - b_0)$ are modeled as error
  - TLS problem: $[(A_{partial} | b_0) + (\delta A_k | \delta b_k)]\ (y_k;-1) = 0$
- TLS algorithm heavily relies on SVD decomposition
  - Singular values and right side eigenvectors
- The locations of ΔA_k are known
- Low cost for partial update of right side eigenvectors and eigenvalues
- The error between y_k and x_k is well bound with a good choice of δA_k and δb_k

Challenges to Efficient Algorithm

- Non-convexity → Sequential convexification
- Nonlinearity → Partial update via TLS
- Large-scale network
  - Bus related variables:

    Real and imaginary components of voltage
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    - Real power generation
    - Reactive power generation
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Plan for Finding the Global Solution

- Global solution using trust region method with primal-dual interior point method
  - Stopping criterion for global solution (Sorensen)
  - Starting point independence
- BARON software package for comparison
  - Widely used and efficient global optimization solver for operation engineering problems
  - Branch-and-bound method

Challenges to Efficient Algorithm

- Non-convexity → Iterative convexification
- Nonlinearity → Partial update via TLS
- Large-scale network
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Variable for Representing Voltages

Key observations:

- Voltages at some buses vary in a consistent way
- $\Phi = [\Phi_1\ \Phi_2;\ -\Phi_2\ \Phi_1]$, $\Phi_1^{\mathrm{T}} = \Phi_1$, $\Phi_2^{\mathrm{T}} = -\Phi_2$
- Rank of Φ is ***always*** 4 regardless of a system

At the i^{th} bus, define

- α_i (4×1) as $\phi_i^T v$ where ϕ_i is the eigenvectors of Φ corresponding to nonzero eigenvalues
- α_i^0 ($2N_B$-4×1) as $(\phi_i^0)^T v$ where ϕ_i^0 is the eigenvectors with zero eigenvalues spanning the null space of Φ

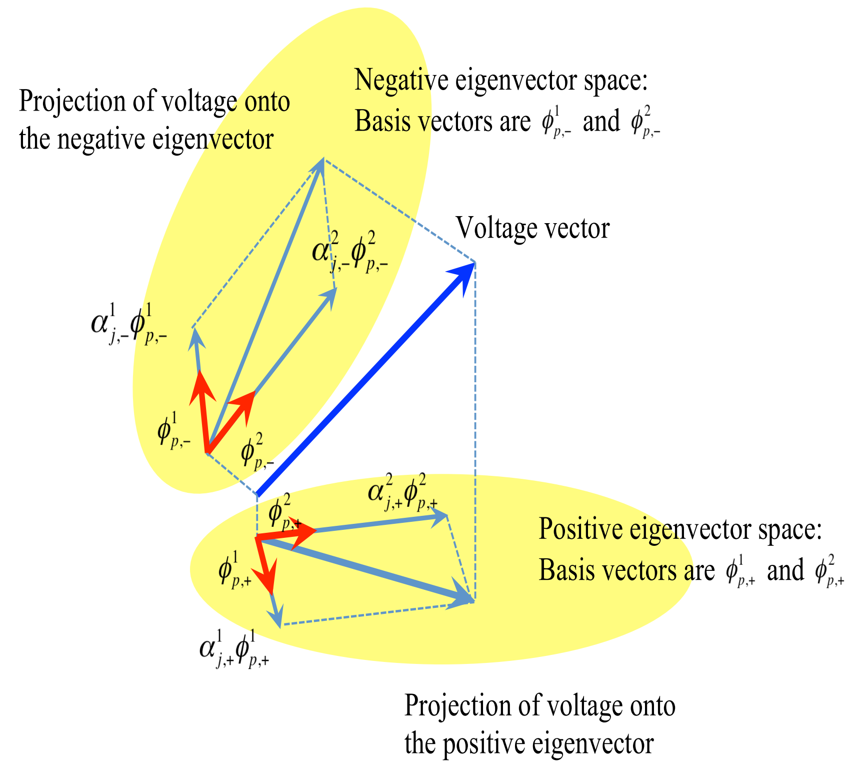
$\rightarrow (\alpha_i,;\ \alpha_i^0) = (\phi_i,\ \phi_i^0)^T\ v$

- Voltage v is reconstructed: $v = \phi_i\alpha_i\ + \phi_i^0\alpha_i^0$

Subspace Problems

Idea

- Power balance equations at Bus *j* yields $p_j = p_j(\alpha_u)$, $q_j = q_j(\alpha_u)$ if ϕ_u is not in the null space of Φ_j
- All the variables → α
- Make local decisions on multiple subspaces
- Adjust the results globally
- Sparsity needs to be preserved in each sub-problem



Parallel Algorithm

- Reformulate AC OPF problem with α_u and α_u^0 by dropping voltage and generator variables
- Fix the values for α_u^0 with respect to v_{k-1}
- Number of variables in the subspace problem:

  $\underline{4} \ll 3N_G + 2N_B \sim \vartheta(N_B)$
- Computation of sub optimization problem
  - Low cost to solve a small problem
  - Can utilize parallel computation
- Central adjustment of α's

Subspace Problem

Branch-and-bound AC OPF sub-problem

$$\min_{\alpha_u} \left\{ \alpha_u^T \left(\sum_y \phi_u^k c_y^R \Phi_y^p \phi_u \right) \alpha_u + 2 \left[\sum_y (\phi_u)^T c_y^R \Phi_y^p \phi_u^0 \alpha_u^0 \right]^T \alpha_u \right\}$$

$$s.t. \begin{cases} \alpha_u^T \Gamma_{uj}^p \alpha_u + 2\overline{a_{uj}} \alpha_u + \overline{d_{uj}^p} = 0, \forall\, j \in PQ \\ :\Gamma_{uj}^p = \phi_u^T \Phi_j^p \phi_u, \overline{a_{uj}} = \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T \Phi_j^p \phi_u, \overline{d_{uj}^p} = d_j^p + \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T \Phi_j^p \phi_u^0 \alpha_u^0 \\ \alpha_u^T \Gamma_{uj}^q \alpha_u + 2\overline{b_{uj}} \alpha_u + \overline{d_{uj}^q} = 0, \forall\, j \in PQ \\ :\Gamma_{uj}^q = \phi_u^T \Phi_j^q \phi_u, \overline{b_{uj}} = \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T \Phi_j^q \phi_u, \overline{d_{uj}^q} = d_j^q + \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T \Phi_j^q \phi_u^0 \alpha_u^0 \\ \underline{|v_j|}^2 - |v_j^0|^2 \le \alpha_u^T \Gamma_{uj}^w \alpha_u + 2\overline{w_{uj}} \alpha_u \le \overline{|v_j|}^2 - |v_j^0|^2, \ \forall\, j \\ :\Gamma_{uj}^w = \phi_u^T W_j \phi_u, \overline{w_{uj}} = \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T W_j \phi_u, |v_j^0|^2 = \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T W_j \phi_u^0 \alpha_u^0 \\ \alpha_u^T \Gamma_{umj}^f \alpha_u + 2 f_{umj}^f \alpha_u \le \dfrac{f_m}{\left|v^{mf}\right|_2^2} - \left(f_{umj}^f\right)^0, \ \forall\, m \in \{G_u\} \\ \alpha_u^T \Gamma_{umj}^t \alpha_u + 2 f_{umj}^t \alpha_u \le \dfrac{f_m}{\left|v^{mt}\right|_2^2} - \left(f_{umj}^t\right)^0, \ \forall\, m \in \{G_u\} \\ :\Gamma_{umj}^f = \phi_u^T \Pi_f^m \phi_u, \Gamma_{umj}^t = \phi_u^T \Pi_t^m \phi_u, \\ \quad f_{umj}^f = \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T \Pi_f^m \phi_u, \quad f_{umj}^t = \left(\alpha_u^0\right)^T \left(\phi_u^0\right)^T \Pi_t^m \phi_u \\ \underline{p_y^R} - \overline{d_{uj}^p} \le \alpha_u^T \Gamma_{uj}^p \alpha_u + 2\overline{a_{uj}} \alpha_u \le \overline{p_y^R} - \overline{d_{uj}^p}, \ \forall\, y \text{ and offer block } R \\ \underline{q} - \overline{d_{uj}^q} \le \alpha_u^T \Gamma_{uj}^q \alpha_u + 2\overline{b_{uj}} \alpha_u \le \bar{q} - \overline{d_{uj}^q}, \ \forall\, y \end{cases}$$

Convexified sub-problem

$$\min_{\alpha_{uk}} \left\{ \alpha_{uk}^T \left(\sum_y \phi_u^k c_y^R \Phi_y^p \phi_u \right) \alpha_{uk} + 2 \left[\sum_y (\phi_u)^T c_y^R \Phi_y^p \phi_u^0 \alpha_{u(k-1)}^0 \right]^T \alpha_{uk} \right\}$$

$$s.t. \begin{cases} \alpha_{uk}^T \Gamma_{uj}^{p+} \alpha_{uk} + 2\left[\overline{a_{uj}} - \left(\alpha_{u(k-1)}^0\right)^T \Gamma_{uj}^{p-}\right] \alpha_{uk} + \overline{d_{uj}^p} + \left(\alpha_{u(k-1)}^0\right)^T \Gamma_{uj}^{p-} \alpha_{u(k-1)}^0 = 0, \forall\, j \in PQ \\ \alpha_{uk}^T \Gamma_{uj}^{q+} \alpha_{uk} + 2\left[\overline{b_{uj}} - \left(\alpha_{u(k-1)}^0\right)^T \Gamma_{uj}^{q-}\right] \alpha_{uk} + \overline{d_{uj}^q} + \left(\alpha_{u(k-1)}^0\right)^T \Gamma_{uj}^{q-} \alpha_{u(k-1)}^0 = 0, \forall\, j \in PQ \\ :\Gamma_{uj}^p = \Gamma_{uj}^{p+} - \Gamma_{uj}^{p-}, \ \Gamma_{uj}^q = \Gamma_{uj}^{q+} - \Gamma_{uj}^{q-} \\ \alpha_{uk}^T \Gamma_{uj}^w \alpha_{uk} + 2\overline{w_{uj}} \alpha_{uk} \le \overline{|v_j|}^2 - |v_j^0|^2, \ \forall\, j \\ \dfrac{1}{2}\left(\underline{|v_j|}^2 - \overline{\overline{|v_j^0|^2}}\right) \le \left[\overline{w_{uj}} + \left(\alpha_{uk}^0\right)^T \Gamma_{uj}^w\right] \alpha_{uk}; \overline{\overline{|v_j^0|^2}} = |v_j^0|^2 + \left(\alpha_{uk}^0\right)^T \Gamma_{uj}^w \alpha_{uk}^0 \ \ \forall\, j \\ \alpha_{uk}^T \overline{\Gamma_{umj}^f} \alpha_{uk} + 2\overline{f_{umj}^f} \alpha_{uk} \le f_m \left[\dfrac{\left|v_{k-1}^{m,f}\right|_2^2}{v_{ref}^2} + \dfrac{v_{ref}^2}{\left|v_{k-1}^{m,f}\right|_2^2} \right] - \left(f_{umj}^f\right)^0, \ \forall\, m \in \{G_u\} \\ \alpha_{uk}^T \overline{\Gamma_{umj}^t} \alpha_{uk} + 2\overline{f_{umj}^t} \alpha_{uk} \le f_m \left[\dfrac{\left|v_{k-1}^{m,t}\right|_2^2}{v_{ref}^2} + \dfrac{v_{ref}^2}{\left|v_{k-1}^{m,t}\right|_2^2} \right] - \left(f_{umj}^t\right)^0, \ \forall\, m \in \{G_u\} \\ :\overline{\Gamma_{umj}^f} = \phi_u^T \overline{\Pi_f^m} \phi_u, \overline{\Gamma_{umj}^t} = \phi_u^T \overline{\Pi_t^m} \phi_u, \\ \quad \overline{f_{umj}^f} = \left(\alpha_{u(k-1)}^0\right)^T \left(\phi_u^0\right)^T \overline{\Pi_f^m} \phi_u, \quad \overline{f_{umj}^t} = \left(\alpha_{u(k-1)}^0\right)^T \left(\phi_u^0\right)^T \overline{\Pi_t^m} \phi_u, \\ \underline{p_y^R} - \overline{d_{uj}^p} \le \alpha_{uk}^T \Gamma_{uj}^p \alpha_{uk} + 2\overline{a_{uj}} \alpha_{uk} \le \overline{p_y^R} - \overline{d_{uj}^p}, \ \forall\, y \text{ and offer block } R \\ \underline{q} - \overline{d_{uj}^q} \le \alpha_{uk}^T \Gamma_{uj}^q \alpha_{uk} + 2\overline{b_{uj}} \alpha_{uk} \le \bar{q} - \overline{d_{uj}^q}, \ \forall\, y \end{cases}$$

Number of Sub-problems

- Number of possible α: N_B
  - Each bus, Φ_j^p and Φ_j^q share null space
  - ϕ_j is uniquely defined → only ***<u>one</u>*** set of α
- Bus $\hat{j}$: directly connected with neither PV nor the reference buses
  - $\phi_{\hat{j}}$ lies the null space of Φ_j^p and Φ_j^q if a generator is located at j
  - $\alpha_{\hat{j}}$: not appear in the power balance, isolated with *p, q, and* ξ

  → *$\hat{j}$ is not a suitable choice for the subspace problem*

  → *Exclude such subspaces*
- Number of the proper choice for the subspace = τN_G
  - ω (= N_L/N_B) is a good estimator for τ: WECC ~ 2.5, EI ~ 3.5
  - τ is approximately constant for various IEEE model systems ~ 2

N_B	9	14	30	118	300	2746
N_G	3	5	6	54	69	520
τN_G	6	9	16	91	143	983

Central Adjust of Local Solutions

$$\min_{v_k, p_k, q_k} \sum_{j=1}^{\tau N_G} \left[\left\| \alpha_j - \left(\phi_j^p\right)^T v_k \right\|_2^2 \right] + \lambda \left\| v_k - v_{k-1} \right\|_2^2$$

- λ is the smallest nonzero eigenvalue of $\Sigma_j\, \phi_j^p (\phi_j^p)^T$
- τ is the number of buses that are:
  PV or ref bus, OR, directly connected with them
- Unconstrained quadratic programming
  - ϕ_j^p are all known and unchanged
  - $v_k = v^* = E(D + \lambda I)^{-1} E^T (\Sigma_j\, \phi_j^p \alpha_j + \lambda v_{k-1})$
    where $\Sigma_j\, \phi_j^p (\phi_j^p)^T = EDE^T$
  - Heuristic approach: $v_k = v_{k-1} + \Upsilon_k v^*$

Challenges to Efficient Algorithm

- Non-convexity → Sequential convexification
- Nonlinearity → Partial update via TLS
- Large-scale network → Sequential subspace optimization with parallelization

Plan for an Efficient Algorithm

- Parallel computation
  - Building a supercomputer
  - Developing a parallel algorithm to fully utilize multiple core processors
- Each core solves a small optimization problem
  - Number of variable is constant
  - Number of subspace problem increases in $\vartheta(N_G)$
  - Tests on large-scale systems

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